UDC 538.4:534.222.2

THE STRUCTURE OF ELECTROHYDRODYNAMIC SHOCK WAVES

PMM Vol. 36, №5, 1972, pp. 851-865

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(Received February 25, 1972)

The structure of electrohydrodynamic shock waves is analyzed in the cases of small Prandtl numbers, when the medium temperature can be considered constant, and of considerable Prandtl numbers, when heat conduction processes can be neglected. The interaction parameter is taken to be arbitrary.

It is shown that, when the electric field and the velocity components E_I^* and u^*_I normal to the shock wave front are of the same sign (the product $u_I^* E_I^* > 0$; throughout the following analysis $u_I^* > 0$) is assumed), the shock wave has always a structure and the electric field component normal to its front is continuous along the shock wave. If the product $u_I^* E_I^* < 0$ and the current density $j^* > 0$, a shock wave structure does not occur for all values of parameters ahead of the shock wave. Analysis of the structure, when the latter exists, shows that behind the shock wave front the electric field is either equal to that ahead of it ($E_{II}^* = E_I^*$) or is related to velocity by the equation $u^*_{II} + bE^*_{II} = 0$. The regions of parameter variation ahead of the wave front, in which one or the other of these cases exist, are determined. These relationships make it possible to close the system of equations which define the state at the electrohydrodynamic shock wave front, as derived in [1].

The class of evolutionary shock waves which have no structure is indicated. It follows from the analysis of shock wave structure and their evolution that in electrohydrodynamics such waves are always of the compression kind.

If the velocity and the electric field behind the wave front satisfy the equation $u_{II}^* + bE_{II}^* = 0$, the system of equations at the shock wave front can be reduced to a cubic equation with respect to the velocity behind the wave front. When the smallest of the three possible real roots of the latter is greater than the speed of sound behind the wave front, the shock wave has no structure and is nonevolutionary. The range of parameters ahead of the wave front for which the wave has a structure is defined for the case in which the smallest real root does not exceed the speed of sound. The other two roots of the cubic equation relate to nonevolutionary shock waves which have no structure. If the cubic equation has only a single real root, the shock wave has always a structure, provided that it is less than the speed of sound behind the wave front (the wave is evolutionary), and has no structure in the opposite case.

Analysis of the shock wave structure shows that, when $u_I^* E_I^* < 0$, the current density $j^* < 0$, and $u_I^* + bE_I^* \neq 0$, the electric field at the shock wave front is continuous. If, however, $u_I^* + bE_I^* = 0$, this field may be discontinuous. In such case for the determination of parameters behind the wave front it is necessary to specify the electric field normal component E_{II}^*

(or any other parameter) behind the front. The range within which E_{11}^* can be specified is indicated. If at the same time $u_{11}^* + bE_{11}^* \neq 0$, the velocity ahead of the front of such shock waves is greater than the speed of sound and behind it, lower than the latter. If $u_{11}^* + bE_{11}^* = 0$, the velocity of the medium in front and behind the wave front is supersonic. All waves of this kind have a structure and are evolutionary.

1. Statement of the problem. Let us consider the stationary flow of gas with a positive bulk charge in an electric field in the electrohydrodynamic approximation [2]. We direct the x-axis along the stream velocity, and assume that the electric field has only a component parallel to that axis and that all parameters of flow depend only on x. In a dimensionless form the equations defining such flow with allowance for viscosity and heat conduction can be written as [1]

$$\rho u = 1, \quad l\frac{du}{d\zeta} = u + \frac{1}{\gamma M_1^2}\frac{T}{u} - SE^2 - \Pi, \quad \Pi = \text{const} \tag{1.1}$$

$$\frac{3l}{4(\gamma-1)M_1^2 \Pr}\frac{dT}{d\zeta} + lu\frac{du}{d\zeta} = \frac{1}{(\gamma-1)M_1^2}T + \frac{1}{2}u^2 + 2SJ(\varphi - \varphi_1) - \Sigma, \quad p = \rho T, \quad \Sigma = \text{const} \tag{1.2}$$

$$\frac{dE}{d\zeta} = \frac{\delta}{E + R_q u}, \quad \frac{d\varphi}{d\zeta} = -QE, \quad R_q J = q(E + R_q u), \quad J = \text{const} \tag{1.3}$$

The dimensionless parameters appearing in Eqs. (1.1) – (1.3) are defined by formulas

$$\begin{gathered}
\rho = \frac{\rho^*}{\rho_1^*}, \quad u = \frac{u^*}{u_1^*}, \quad p = \frac{p^*}{p_1^*}, \quad T = \frac{T^*}{T_1^*}, \quad E = \frac{E^*}{E_1^*} \\
q = \frac{q^*}{q_1^*}, \quad \varphi = \frac{4\pi q_1^* \varphi^*}{E_1^{*2}}, \quad M_1 = \frac{u_1^*}{a_1^*}, \quad a_1^{*2} = \frac{\gamma p_1^*}{\rho_1^*} \\
\gamma = \frac{c_p}{c_v}, \quad S = \frac{E_1^{*2}}{8\pi\rho_1^* u_1^{*2}}, \quad Q = \frac{4\pi q_1^* L}{E_1^*} \\
J = \frac{j_1^*}{q_1^* u_1^*}, \quad R_q = \frac{u_1^*}{bE_1^*}, \quad Pr = \frac{c_p \eta}{\varkappa} \\
\delta = R_q Q J = \frac{4\pi j_1^* L}{bE_1^{*2}}, \quad \zeta = \frac{x}{L}, \quad l = \frac{\lambda}{L}, \quad \lambda = \frac{4\eta}{3\rho_1^* u_1^*}
\end{gathered} \tag{1.4}$$

where ρ^*, u^*, p^* and T^* are the dimensional density, velocity, pressure, and temperature of the medium, respectively; q^*, j^*, E^* and φ^* are, respectively, the bulk charge density, the current density, the electric field intensity, and the electric potential; b, η and $\varkappa$ are the coefficients of mobility, viscosity, and thermal conductivity, respectively, which in the subsequent analysis will be considered constant; c_p and c_v are the specific heats, and L is a quantity having the dimension of length.

The subscript 1 denotes parameters at a particular point of the stream at which the terms appearing in the left-hand sides of the second of Eqs. (1.1) and of the first of Eqs. (1.2) and related to viscosity and thermal conductivity can be neglected. This means that at that point the effect of the stream momentum produced by viscosity and that of the heat flux density produced by heat conduction and the work of viscous forces on the total density of the stream momentum and of the energy flux are negligibly small. In this case the constants of integration Π and Σ defined by flow parameters at the point are

$$\Pi = 1 + \frac{1}{\gamma M_1^2} - S, \qquad \Sigma = \frac{1}{2} + \frac{1}{(\gamma - 1) M_1^2}$$

In the stated approximation the constants Π and Σ coincide with the dimensionless densities of the momentum and gasdynamic energy fluxes in an inviscid and nonheat-conducting flow.

In the following we propose to investigate the structure of the electrohydrodynamic shock wave. For this it is necessary to determine the behavior of integral curves of Eqs. (1.1) - (1.3). Two cases of Prandtl numbers $\Pr \ll 1$ and $\Pr \gg 1$ will be considered. The first corresponds to a high thermal conductivity coefficient of the medium, when $T = 1$ can be substituted for the first of Eqs. (1.1). The second relates to the case of a low thermal conductivity coefficient, when the term containing $dT/d\zeta$ in the first of Eqs. (1.2) can be neglected.

It was shown in [1] that in electrohydrodynamics the system of equations at the shock wave is not closed, owing to the lack of equations defining the component of the electric field E_{II}^* normal to the discontinuity front behind the shock wave. Here and subsequently subscripts I and II denote, respectively, parameters ahead and behind the shock wave front. The missing equation was derived in [1] on the assumption of intrinsic exsistence of shock wave structure and of monotonically decreasing velocity within the shock wave structure. This assumption is always satisfied, e. g. for small interaction parameters. It was, also, shown in [1] that for $u^* > 0$ and $j^* > 0$, when ahead of the wave front the electric field $E_I^* > 0$, we have $E_{II}^* = E_I^*$. If $E_I^* < 0$ (the gas flows counter to the electric field), then two cases are possible: either $E_{II}^* = E_I^*$ (when $u_{II}^* + bE_I^* > 0$) or $E_{II}^* = -u_{II}^* / b$ (when $u_{II}^* + bE_I^* < 0$).

The missing relationship at the shock wave front in the case of an arbitrary interaction parameter will be determined by analyzing the shock wave structure.

2. Shock wave structure in a gas with high thermal conductivity coefficient. Let us examine the behavior of integral curves of Eqs. (1.1), (1.3), and $T = 1$ in the half-plane uE with $u > 0$. Dividing the first of Eqs. (1.3) by the second of (1.1), we obtain

$$\frac{dE}{du} = \frac{\varepsilon}{(E + R_q u)(u + M_t^{-2}u^{-1} - SE^2 - \Pi_t)}, \qquad \varepsilon = \delta l$$

$$M_t = u_1^* a_t^{*-1}, \quad \Pi_t = 1 + M_t^{-2} - S, \quad a_t^{*2} = p_1^* \rho_1^{*-1} \tag{2.1}$$

where a_t^* is the isothermic speed of sound, $q^* > 0$ and $u_1^* > 0$. Let the current $j_1^* > 0$, the field $E_1^* < 0$ and $M_t > 1$ (the case of $j_1^* < 0$ is considered below), with the electric Reynolds number $R_q < 0$ and parameters $Q < 0$ and $\varepsilon > 0$. The condition $J = 1 + R_q^{-1} > 0$ implies that $|R_q| > 1$, hence $J \lesssim 1$. The order of magnitude of parameter λ is that of the free path. It will be readily seen that $\varepsilon = 4\pi q_1^* \lambda R_q J / E_1^* \ll 1$.

We introduce the notation

$$L_1 = E + R_q u, \qquad L_2 = u + \frac{1}{M_t^2 u} - SE^2 - \Pi_t \tag{2.2}$$

Let us plot in the considered half-plane the lines $L_1 = 0$ and $L_2 = 0$ (Figs. 1 - 4) which we shall henceforth denote by L_1° and L_2°, respectively. The disposition of these lines is determined by parameters M_t, R_q and S. It is not difficult to see that the disposition of lines L_1° and L_2° in the dimensional plane u^*E^* depends on parameters b, m^*, Π^* and a_t^*, with $m^* = \rho_1^* u_1^*$ and $\Pi^* = m^* u_1^* + a_t^{*2} \frac{m^*}{u_1^*} - \frac{1}{8\pi} E_1^{*2}$.

Along lines L_1° and L_2° the integral curves of Eq. (2.1) have vertical tangents. Line L_2° has a vertical asymptote $u = 0$ and passes through points $u = 1$ and $E = \pm 1$. When $u \to \infty$, the field $E \sim \pm (u / S)^{1/2}$ along line L_2°. Depending on the value of parameters S and M_t line L_2° can have various patterns. For $S < (1 - M_t^{-1})^2$ it consists of two branches which intersect axis u at points

$$u_{1,2}^\circ = \frac{1}{2}\left(1 + \frac{1}{M_t^2} - S\right) \mp \left[\frac{1}{4}\left(1 + \frac{1}{M_t^2} - S\right)^2 - \frac{1}{M_t^2}\right]^{1/2} \tag{2.3}$$

At these points line L_2° has vertical tangents. For $S \ll 1$ parameters $u_1^\circ = 1 / M_t^2$ and $u_2^\circ = 1$. When $S = (1 - M_t^{-1})^2$, the two branches of curve L_2° join at point $u_{1,2}^\circ = 1 / M_t$ which is also the point of intersection of line L_2° with the u-axis. With increasing interaction parameter S ($S > (1 - M_t^{-1})^2$) the branches of curve L_2° become separated again, with one branch lying entirely in the upper half-plane $u > 0$, $E > 0$, while the second (symmetric) in the lower half-plane $u > 0, E < 0$. Both branches have their extrema at points

$$u_m = \frac{1}{M_t}, \quad E_m = \pm\left(1 - \frac{S^*}{S}\right)^{1/2}, \quad S^* = \left(1 - \frac{1}{M_t}\right)^2 \tag{2.4}$$

Note that along the straight line $u = u_m$ the flow velocity is equal to the speed of sound, while for $u < u_m$ and $u > u_m$ it is, respectively, subsonic and supersonic. Hence, for $S < S^*$ the left-hand branch of line L_2° lies in the subsonic region and its right-hand branch in the supersonic region.

Lines L_1° and L_2° may intersect either at one or three points, whose coordinates $u^{(i)}$ and $E^{(i)}$ $(i = 1, 2, 3)$ are determined by the equations $E = -R_q u$ and

$$SR_q^2 u^3 - u^2 + \left(1 + \frac{1}{M_t^2} - S\right) u - \frac{1}{M_t^2} = 0 \tag{2.5}$$

If $S \leqslant S^*$, there is always a point of intersection in the subsonic region, as can be seen in Fig. 1. When $S > S^*$, either a single point lying in the subsonic (Fig. 2) or the supersonic region (Fig. 3), or three points of intersection are possible. In the latter case either all three intersection points lie in the supersonic region ot two are in the supersonic and one is in the subsonic region (Fig. 4). Let us assume that $S > S^*$ and that the curves L_1° and L_2° intersect at the point of minimum of the upper branch of curve L_2°, i.e. that one of the roots of (2.5) is $u^{(1)} = u_m = M_t^{-1}$. Obviously there can be no other points of intersection of L_1° and L_2° to the left of line $u = u_m$. Substituting this expression for $u^{(1)}$ into (2.5), we obtain

$$S = \frac{(M_t - 1)^2}{M_t^2 - R_q^2} \tag{2.6}$$

It follows from this that $R_q^2 < M_t^2$, since $S > 0$. The remaining two roots $u^{(2)}$ and $u^{(3)}$ are defined by the equation

$$u^2 + \frac{SR_q^2 - M_t}{SR_q^2 M_t} u + \frac{1}{SR_q^2 M_t} = 0 \tag{2.7}$$

The quadratic trinomial (2.7) has positive real roots in the interval $u_m < u < 1$, if the inequalities $SR_q^2 / M_t \leqslant 3 - 2\sqrt{2}$, $SR_q^2 M_t > 1$ and $R_q^2 > 1$. With the use of (2.6) the first two of these inequalities can be written as

$$\frac{M_t^2}{M_t(M_t-1)^2+1} < R_q^2 \leqslant \frac{(3-2\sqrt{2})M_t^3}{(M_t-1)^2+(3-2\sqrt{2})M_t} \tag{2.8}$$

For $M_t > \sqrt{2}+1$ the fractions which bound R_q^2 from above and below are, respectively, greater and smaller than unity, hence there is a nonempty set of values of parameters M_t and R_q which satisfy the conditions $R_q^2 > 1$ and $R_q^2 < M_t^2$ under which the inequalities (2.8) are satisfied.

It has, thus, been shown that there is a range of parameters $S > 0$, $|R_q| > 1$ and $M_t > 1$ within which in the interval $0 < u < 1$ there exist three points of intersection of curves L_1° and L_2°, which correspond to three real roots of Eq. (2.5). Two of these are always in the supersonic region of velocity u, while the third may lie in the super- or subsonic region. This proves the existence of a range of parameters m^*, Π^*, a_t^* and b within which curves L_1° and L_2° have three intersection points in the dimensional plane u^*E^*. If $|R_q| \geqslant M_t$, then for any interaction parameter S curves L_1° and L_2° have always, in the interval $0 < u < 1$, only a single intersection point in the subsonic region of velocity u. The condition $|R_q| \geqslant M_t$ implies that the diffusion velocity at point $x = x_1$, which is equal to the difference of velocities of charged and neutral particles, is lower or equal to the speed of sound at that point.

Note that Eq. (2.5) is valid for the determination of the velocity of gas behind the shock wave in investigation of relationships at discontinuities in electrohydrodynamics, when the electric field normal component behind the wave is such that $E_{II}^* = -u_{II}^*/b$, and $T^* = \text{const}$ [1]. In fact, at constant gas temperature $p = \rho$; eliminating with the aid of this equality pressure p from the first of formulas (6.2) in [1], we obtain formula (2.5). Thus the velocity u and the electric field intensity E at points of intersection of L_1° and L_2° make it possible to determine the state of gas behind an electrohydrodynamic shock wave. The selection of one intersection point, if three of these exist, is carried out by analyzing the behavior of integral curves in the problem of shock wave structure.

Let us assume that $E > 0$ and investigate the behavior of integral curves, when L_1° and L_2° have a single intersection point and are of the form shown in Figs. 1 and 2. The first of Eqs. (1.3) implies that the dimensionless electric field always decreases with increasing ζ (in the downstream direction). This makes it possible to select the direction of motion along the integral curves, as shown by arrows in Figs. 1 – 4. The integral curves which define flows in the presence of positive current J always lie below curve L_1°. Since the parameter $\varepsilon \ll 1$, hence the slope of integral curves at some distance from curves L_1° and L_2° is small. In proximity to these curves, when $L_1 \sim \varepsilon$ or $L_2 \sim \varepsilon$, the tangent of the angle of inclination of integral curves is $dE/du \sim 1$, the curves undergo an abrupt turn, and then run along lines L_1° and L_2° in their ε-neighborhood. With decreasing parameter ε (the viscosity coefficient $\eta \to 0$) the integral curves approach closer and closer lines L_1° and L_2°, and at the limit $\varepsilon \to 0$ merge with the latter and follow these from thereon. The angle of inclination of integral curves away from lines L_1° and L_2° tends to zero, when $\varepsilon \to 0$. Analysis of the second of Eqs. (1.1) and of the first of Eqs. (1.3) shows that various sections of integral curves in the uE-plane relate to various flow patterns in the physical plane.

Let us consider the integral curve I which for $\varepsilon \to 0$ emerges horizontally from a small neighborhood of point $u = 1$, $E = 1$. Above this point curve I runs along line L_2° at close proximity to it. Let $\Delta\zeta_u$ and $\Delta\zeta_E$ be, by definition, characteristic distances at which the velocity and the electric field determined, respectively, by the second of Eqs. (1.1) and the first of Eqs. (1.3), vary by an order of magnitude of unity. Along the part of integral curve I where $u^{II} < u < u^{I}$ the distances $\Delta\zeta_u \sim 1$ and $\Delta\zeta_E \sim 1$, since there $|L_2| \sim l \sim \varepsilon$ and $|L_1| \sim 1$. A region of varying flow velocity and electric field intensity, in which the effect of viscosity on parameter variation is small, corresponds in the physical field to this part of curve I. Since along the latter $|L_1| \sim 1$ and $|L_2| \sim 1$, for $u^{III} < u < u^{II}$, we have $\Delta\zeta_u \sim \varepsilon \ll 1$ and $\Delta\xi_E \sim 1$. A narrow flow region with high rate of velocity variation and virtually constant electric field corresponds in the physical field to this part of curve I. Finally, along the part of integral curve I where $u^{IV} < u < u^{III}$ the distances $\Delta\zeta_u \sim \varepsilon$ and $\Delta\zeta_E \sim \varepsilon$, since there $|L_1| \sim \varepsilon$ and $|L_2| \sim 1$. A narrow region of flow with high rate of velocity and electric field variation corresponds in the physical plane to this part of the curve. In the neighborhood of the point of intersection of lines L_1° and L_2° defined by coordinates $u^{(1)}$ and $E^{(1)}$, $u^{(1)}$ is the root of Eq. (2.5) and the integral curve I has a vertical tangent at its intersection with L_2° and then continues along the latter. Along this part of the integral curve $\Delta\zeta_u \sim 1$ and $\Delta\zeta_E \sim 1$, since there $|L_1| \sim 1$ and $L_2 \sim \varepsilon$. A region of flow with varying u and E in which viscosity has little effect on parameter variation corresponds in the physical plane to this part of the integral curve. Let us assume that part of integral curve I where $u > 1$ and $E > 1$ relates to the flow of gas ahead of the shock wave, while its part where $E < E^{(1)}$ relates to the flow behind it. Along these parts of the integral curve the contribution of the stream momentum density to the over-all density of the momentum flow can be neglected. The part of the integral curve, along which $E^{(1)} < E < 1$, defines the structure of the electrohydrodynamic shock wave. A region of flow with considerable variation of parameters u and E corresponds in the physical plane to this part of the integral curve. The variation of parameters u and E in the physical planes ζu and ζE which corresponds to integral curve I is shown in Fig. 5. At the limit $\varepsilon \to 0$ the integral curve I leaves line L_2° at point $u = 1$, $E = 1$ and joins line L_2° at point $u = u^{(1)}$, $E = E^{(1)}$. These points correspond, respectively, to the state of gas immediately ahead of and behind the shock wave front. Parameters $u^{(1)}$ and $E^{(1)}$ are related by the expression $E^{(1)} + R_q u^{(1)} = 0$, which in dimensional form can be written as

$$u_{II}^* + bE_{II}^* = 0 \tag{2.9}$$

Formula (2.9), which was derived by analyzing the shock wave structure for arbitrary interaction parameter S, is the missing relationship which closes the system of equations defining the state at the shock wave front derived in [1].

It will be seen from Figs. 1 and 2 that many integral curves lie in the vicinity of point $u^{(1)}$, $E^{(1)}$ all of which pass through this point and, when $\varepsilon \to 0$, leave line L_2° at points lying above and below point $u = 1$, $E = 1$. To every of these points corresponds a particular state ahead of the shock wave front whose structure is defined by the integral curve emanating from that point. The integral curve which for $\varepsilon \to 0$ coincides with the segment of the straight line $E = E^{(1)}$ is denoted by the numeral II. Integral curves which lie above line II define the structure of shock waves with a discontinuity of the electric field at their front. The integral curve II itself and all curves lying below it, including those for $E < 0$, define the structure of shock waves along which for $\varepsilon \to 0$ the electric field is continuous. Those points at which integral curves join line L_2' relate to the state immediately behind the wave front. As shown in [1], the equations of gasdynamic parameters behind the wave front for a continuous field are the same as the related equations of conventional gasdynamics. Note that for $E < 0$ the dimensional electric field corresponding to integral curves leaving L_2° is positive ahead of the wave front.

If the pattern of line L_2° coincides with that shown in Fig. 1 ($S < S^*$), the gas velocity behind the shock wave front is subsonic and the shock wave is evolutionary [1]. If, however, $S > S^*$, then point $u^{(1)}$, $E^{(1)}$ of intersection of lines L_1° and L_2° may, for specific values of parameters R_q, M_t and S lie in the supersonic region (Fig. 3). For $E > E^{(1)}$ the integral curves emanating from the ε-neighborhood of line L_2°, run at a small slope up to the intersection with line L_1°, follow its ε-neighborhood up to intersection with line L_2° in the ε-neighborhood of point $u^{(1)}$, $E^{(1)}$, and then turn to continue at a small negative slope in the direction of increasing u. The physical meaning of these parts of integral curves up to intersection with line L_2° is similar to that previously described. Along the part of integral curves beyond the intersection point, where $u > u^{(1)}$ the lengths $\Delta\zeta_u \sim \varepsilon$ and $\Delta\zeta_E \sim 1$, since there $|L_1| \sim 1$ and $L_2 \sim 1$. In the physical plane this part of integral curves defines a flow with abrupt increase of velocity and constant electric field (Fig. 6). Note that for $\varepsilon \to 0$ the derivative $du / d\zeta \to -\infty$ along the integral curves in the region where $L_2 < 0$ (above line L°_2), while in the region in which $L_2 > 0$ (below line L_2°) the derivative $du / d\zeta \to \infty$. Beyond the intersection with line L_2° the integral curves nowhere run along the latter, i.e. there is in this case no section which corresponds to an inviscid flow. Thus, for $E > 0$ there are no integral curves which would define the structure of a shock wave.

Let us now consider the behavior of integral curves, when lines L_1° and L°_2 intersect at three points A $(u^{(1)}, E^{(1)})$, B $(u^{(2)}, E^{(2)})$ and C $(u^{(3)}, E^{(3)})$ with $u^{(1)} < u^{(2)} < u^{(3)}$. Let for $u^{(1)} < u_m$ the pattern of lines L_1° and L_2° be that shown in Fig. 4. The integral curves which leave the ε-neighborhood of line L_2° above it do not yield in region $u > u^{(3)}$ a shock wave structure. This case is analogous to that of a single intersection point of lines L_1° and L_2° lying in the supersonic region.

The integral curves which leave the upper ε-neighborhood of line L_2° in the region of $u_m < u < u^{(2)}$ define the structure of evolutionary shock waves. This is analogous to the previously considered case of a single intersection point of lines L_1° and L_2° lying in the subsonic region. If, at the same time, $u_* < u < u^{(2)}$, where u_* is the greater root of equation $L_2(u, E^{(1)}) = 0$, the condition $u_{II}^* + bE_{II}^* = 0$ must be specified behind the shock wave front. When $u_m < u \leqslant u_*$, then $E_{II}^* = E_I^*$ at the shock wave front. The section of line L_2° lying between points B and C is situated in the region of $J < 0$.

If $u^{(1)} > u_m$, all intersection points of lines L_1° and L_2° lie in the supersonic region and there are no integral curves defining a shock wave structure.

The relationships at electrohydrodynamic shock waves derived in [1] admit an evolutionary discontinuous transition from any arbitrary point of line L_2° to any other arbitrary point of that line with $u^{(1)} < u < u_m$ (for $S > S^*$, Fig. 4), when $u > u^{(3)}$, or with $u^{(1)} < u < u_1^\circ$ (for $S \leqslant S^*$, u_1° is the left-hand intersection point defined by (2.3)). It follows from the present investigation that such shock waves have no structure.

The foregoing analysis dealt with the case of $E > 0$. When $E < 0$, the structure of shock waves exists and the field at their front is continuous.

3. Shock wave structure in a gas with low thermal conductivity coefficient. Let us consider the structure of a shock wave, when $Pr \gg 1$. It will be shown below that along integral curves which define in the uE-plane the flow inside a shock wave the variation of electric potential is small. Neglecting in the first of Eqs. (1.2) the term containing $dT / d\zeta$ and setting $\varphi = \varphi_1$, we obtain with the use of the second of Eqs. (1.1) for the temperature of gas the expression

$$T = \gamma(\gamma - 1) M_1^2 (1/_2 u^2 - SE^2 u - \Pi u + \Sigma) \tag{3.1}$$

With the use of (3.1) we can eliminate temperature from the second of Eqs. (1.1). Let us introduce the notation

$$L_1 = E + R_q u, \qquad L_2 = \frac{\gamma + 1}{2} u + \left(\frac{1}{M_1^2} + \frac{\gamma - 1}{2}\right)\frac{1}{u} - \gamma SE^2 - \gamma\left(1 + \frac{1}{\gamma M_1^2} - S\right) \tag{3.2}$$

It will be readily seen that

$$\frac{du}{d\zeta} = \frac{L_2}{l}, \qquad \frac{dE}{d\zeta} = \frac{\delta}{L_1} \tag{3.3}$$

Dividing the second of Eqs. (3.3) by the first, we obtain

$$\frac{dE}{du} = \frac{\varepsilon}{L_1 L_2}, \qquad \varepsilon = \delta l \ll 1 \tag{3.4}$$

Let us examine the behavior of integral curves in the half-plane uE, $u > 0$. Let $E_1^* < 0$, $q^* > 0$, $j_1^* > 0$ and $M_1 > 1$. We then have $Q < 0$, $R_q < 0$ and $\delta > 0$, as well as $|R_q| > 1$. We denote lines $L_1 = 0$ and $L_2 = 0$ in the half-plane uE, $u > 0$, respectively, by L_1° and L_2°. Along these lines the integral curves defined by Eq. (3.4) have vertical tangents. Line L_2° has a vertical asymptote $u = 0$, and for $u \to \infty$ the electric field along line L_2° is $E \sim [(\gamma + 1)\, u / 2\gamma S]^{1/2}$. We introduce the notation

$$S_* = 1 + \frac{1}{\gamma M} - \frac{\gamma + 1}{\gamma}\left[\frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1) M_1^2}\right]^{1/2}$$

It is shown below that $S_* > 0$ for any $\gamma > 1$ and $M_1 > 1$. When the interaction parameter $S < S_*$, the curve L°_2 consists of two branches which intersect line $E = 0$ at the two points, where their tangents are vertical. For $S = S_*$ these two branches unite at the single intersection point of L_2° with the u-axis. With increasing S branches of line L_2° become again separated, with one lying entirely in the part $u > 0$, $E > 0$ of the considered half-plane, and the other (symmetric to it) in the half-plane $u > 0$, $E < 0$. The two branches have extrema at points

$$u_m = \left[\frac{2}{(\gamma + 1) M_1^2} + \frac{\gamma - 1}{\gamma + 1}\right]^{1/2}, \qquad E_m = \pm\left(1 - \frac{S_*}{S}\right)^{1/2} \tag{3.5}$$

Line L_2° always passes through points $u = 1$, $E = \pm 1$. Since $E_m^2 \leqslant 1$, then $S_* > 0$.

Let us plot in the half-plane uE, $u > 0$ line L_3° along which the velocity is equal to the speed of sound. Along it $u^2 M_1^2 = T$, hence

$$E = \pm \frac{1}{\sqrt{S}}\left(cu + \frac{1}{u}\Sigma - \Pi\right)^{1/2}, \qquad c = \frac{\gamma^2 - \gamma - 2}{2\gamma(\gamma - 1)} \tag{3.6}$$

The form of line L_3° depends primarily on the sign of coefficient c. If $c > 0$ ($\gamma > 2$), line L_3° has two branches and, depending on parameters γ, M_1 and S, these may intersect the u-axis with vertical tangents at two points, or at one point (the two branches merge) or, finally, not intersect the u-axis at all. In the latter case the two branches of line L_3° are symmetric and lie in the upper and lower quarters of the half-plane uE, $u > 0$. Thus for $\gamma > 2$ the form of line L_3^0 is generally similar to that of line L_2°.

If $c < 0$ $(\gamma < 2)$, line L_3° has only a single branch which intersects the u-axis with a vertical tangent. For $c = 0$ $(\gamma = 2)$ the form of line $L^\circ{}_3$ is determined by the sign of parameter Π. When $\Pi > 0$, this line has a single branch (as for $c < 0$) and, when $\Pi < 0$, it is a hyperbola whose branches lie in the upper and lower quarters of the half-plane, respectively.

The relative positions of lines L_2° and L_3° can be different. Their intersection is only possible at points of extremum (3.5) of line L_2°, if these exist, or at the point of merging of branches of the line L_2 (i.e. when $S \geqslant S_*$). If lines L_2° and L_3° intersect, the parts of line L_2° contained in the interval $u < u_m$ are in the subsonic region, those in the interval $u > u_m$ are in the supersonic region, and at the point of their intersection the velocity is sonic. If lines L_2° and L_3° do not intersect $(S < S_*)$, the entire left-hand branch of L_2° is in the subsonic, and its entire right-hand branch in the supersonic region.

The coordinates $u^{(k)}$, $E^{(k)}$ $(k = 1, 2, 3)$ of points of intersection of lines L_1° and L_2° are determined by equations $E = -R_q u$ and

$$SR_q^2u^3 - \frac{\gamma+1}{2\gamma}u^2 + \left(1 + \frac{1}{\gamma M_1^2} - S\right)u - \left(\frac{1}{\gamma M_1^2} + \frac{\gamma-1}{2\gamma}\right) = 0 \qquad (3.7)$$

Equation (3.7) coincides with the equation which defines the state of gas behind the shock wave front in investigations of relationships at a discontinuity, when the electric field behind the front is $E_{II}^* = -u_{II}^* / b$. In fact, by eliminating in Eq. (6.2) the pressure p derived in [1], we obtain (3.7). This equation was investigated in [3] in connection with the analysis of one-dimensional flows of perfect gas in the presence of shock waves.

Let us investigate the possible behavior of lines L_1° and L_2° in the case of their intersection at the minimum point of the upper branch of L_2° $(S > S_*)$. It is obvious that there can be no other intersection point to the left of point u_m. Substituting the root $u^{(i)} = u_m$ into (3.7), we obtain

$$S = S_*(1 - R_q^2u_m^2)^{-1}$$

For the determination of the other two roots of (3.7) we have the equation

$$u^2 + u_m(1 - A)u + Au_m^2 = 0, \qquad A = \frac{\gamma+1}{2\gamma SR_q^2u_m} \qquad (3.8)$$

The quadratic trinomial (3.8) has real roots in the interval $u_m < u < 1$, provided that the inequalities $R_q^2 > 1$ and

$$\frac{u_m}{u_m^3 + u_m^2 - 2u_m + 1} < R_q^2 \leqslant \frac{3 - 2\sqrt{2}}{u_m[u_m^2 - (2\sqrt{2} - 1)u_m + 1]} \qquad (3.9)$$

are satisfied. For $u_m < \sqrt{2} - 1$ the expressions which bound R_q^2 from above and below are, respectively, greater and smaller than unity. It can be readily shown that there exists a nonempty set of values of parameters γ, R_q and M_1 which satisfy conditions $\gamma > 1$, $R_q^2 > 1$ and $R_q^2 < u_m^{-2}$, for which the inequalities (3.9) are satisfied

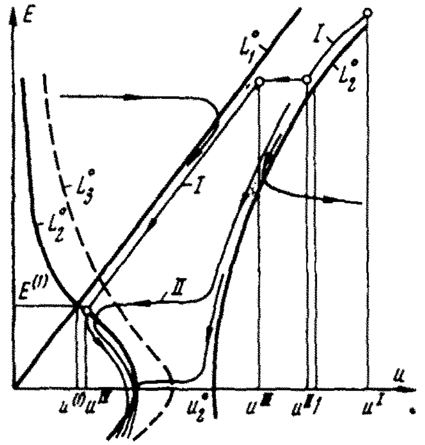

Fig. 1

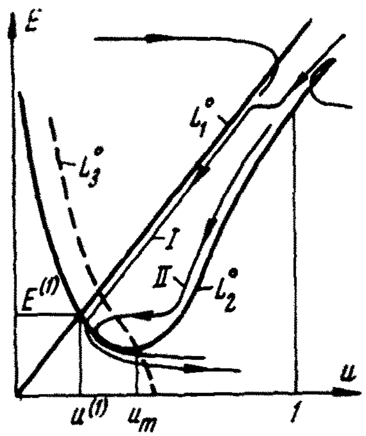

Fig. 2

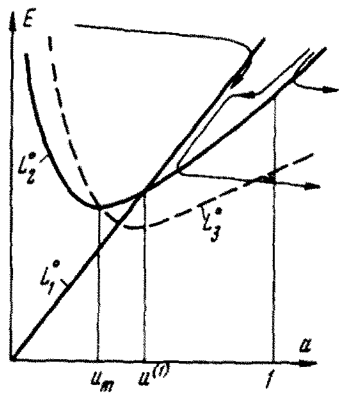

Fig. 3

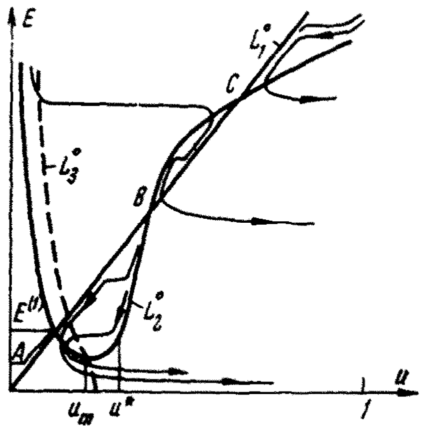

Fig. 4

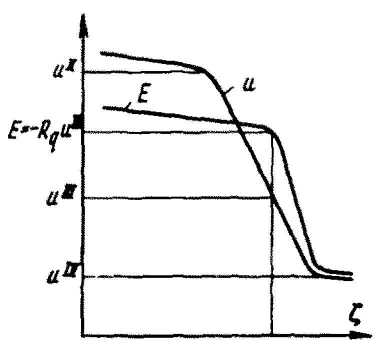

Fig. 5

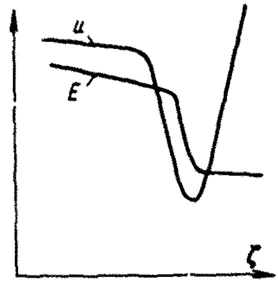

Fig. 6

and, consequently, there are two more intersection points of lines $L_1°$ and $L_2°$, when $u > u_m$. A small variation of parameters, for example that of interaction, which determine the relative position of lines $L_1°$ and $L_2°$ can result in a shift of the intersection point at the minimum of line $L_2°$ either into the subsonic or supersonic region, while the other two intersection points remain in the supersonic region. Thus lines $L_1°$ and $L_2°$ can have three intersection points, two of which are always in the supersonic region and one which can lie either in the sub- or supersonic region. A similar statement concerning the disposition of roots of the polynomial (3.7) was obtained in [3]. When $|R_q| \geqslant u_m^{-1}$, curves $L_1°$ and $L_2°$ in the interval $0 < u \leqslant 1$ always have a single point of intersection in the subsonic region.

The pattern of behavior of integral curves for $\gamma > 1$ is analogous to that considered in Sect. 2 for the case of $T^* = \text{const}$ (Figs. 1 – 4). The dash line $L_3°$ in Figs. 1, 2 and 4 relates to $\gamma < 2$ and in Fig. 3 to $\gamma > 2$.

Let $E > 0$. In this case, if Eq. (3.7) has one real root in the subsonic region, the shock wave has a structure and is evolutionary; behind the discontinuity front the condition $E_{II}^* = E_I^*$ $(E_{II}^* = - u_{II}^* / b)$ is fulfilled, when the shock wave structure is defined by integral curves lying below (above) of integral curve II. Along the latter the electric field $E_{II}^* = E_I^* = - u_{II}^* / b$. When the root of Eq. (3.7) is in the supersonic region or there are three positive real roots in that region (this can only occur for $S > S_*$), there is no shock wave structure. It one of the three real roots (for $S > S_*$) is in the subsonic region (Fig. 4), the shock wave structure exists and the wave is evolutionary, provided that $u_m < u < u^{(2)}$ $(u^{(1)} < u^{(2)} \leqslant u^{(3)}$ are roots of Eq. (3.7)). And if in this case $u_* < u < u^{(2)}$, where u_* is the greater root of equation $L_2(u, E^{(1)}) = 0$, it is necessary to specify $E_{II}^* = - u_{II}^* / b$ behind the shock wave front. If $u_m < u \leqslant u_*$, then at the shock wave front $E_{II}^* = E_I^*$. When the interaction parameter $S < S_*$, one root of Eq. (3.7) is always in the subsonic region and, if in this case there are three intersection points of curves $L_1°$ and $L_2°$, then for $E > 0$ there exist shock wave structures with velocity ahead of the front defined by $u_2° < u < u^{(2)}$ where $u_2°$ is the velocity at the intersection point of the right-hand branch of line $L_2°$ with the u-axis. Such waves are evolutionary. The relationships at electrohydrodynamic shock waves [1] admit for $u > u^{(3)}$ an evolutionary discontinuous transition from any arbitrary point of line $L_2°$ to any other of its points with $u^{(1)} < u < u_m$ (for $S > S_*$) (Fig. 4) or with $u^{(1)} < u < u_1°$ (for $S \leqslant S_*$), where $u_1°$ is the velocity at the intersection point of the left-hand branch of line $L_2°$ with the u-axis. The results of this investigation show that such shock waves have no structure. If $E < 0$, then there exists a shock wave structure and the electric field at the shock wave front is continuous. It is evident that electrohydrodynamic shock waves are always compression waves.

Let us consider the variation of the electric potential along integral curves. It will be seen from the second of Eqs. (1.3) that the characteristic length of potential variation $\Delta\zeta_\varphi$ along the integral curves is everywhere of the order of unity. Hence along the sections of integral curves which define the shock wave structure and to which in the physical plane corresponds the flow region $\Delta\zeta \sim \varepsilon$, the electric potential variation is $\Delta\varphi \sim \varepsilon$. For $\varepsilon \to 0$ parameter $\Delta\varphi \to 0$ and the potential at the shock wave front is continuous. Note that the sections of integral curves to which correspond flow regions $\Delta\zeta \sim 1$ have no physical meaning, since there $\Delta\varphi \sim 1$. Any point of line $L_2°$ in the regions defined above as admissible can correspond to the state immediately ahead of the shock wave front. However, flows ahead of shock front close to perfect are not defined by sections of integral curves lying along line $L_2°$, since in such flows the electric potential varies, while line $L_2°$ has been constructed without allowance for variation of φ. Integral curves which define a close to perfect flow ahead of the wave front lie along the surface of space $uE\varphi$ over which the mass, momentum and energy fluxes of a perfect flow are constant. Close to this surface lie integral curves which define a nearly perfect flow behind the shock wave front.

We stress the difference between the atatement of the electrohydrodynamic shock wave problem and that of the corresponding problem of gasdynamics. In the latter the state ahead and behind the shock wave front is defined by the total mass, momentum and energy fluxes, while in electrohydrodynamics the specification of these fluxes is not sufficient for the determination of parameters ahead of the shock wave front. In the latter case it is necessary to specify parameters u and E ahead of the shock wave. Only then the point of line $L_2°$ which corresponds to the state ahead of the shock wave is uniquely defined.

The presented here analysis of the structure of electrohydrodynamic shock waves makes it possible to indicate for given mass, momentum and gasdynamic energy fluxes the range of flow parameters ahead of the wave front for which evolutionary shock wave structures exist.

4. Shock wave structures with ions moving counter to the gas stream. Let $q^* > 0$, $u^* > 0$, $E_1^* < 0$, $j_1^* < 0$ and $M_1 > 1$ with $\varepsilon < 0$ and $R_q < 0$. From the third of Eqs. (1.3) follows that $|R_q| < 1$. The equation of line $L_2°$ in the case of $\mathrm{Pr} \ll 1$ is given by the second formula of (2.2) and for that of $\mathrm{Pr} \gg 1$ by the second formula of (3.2). It can be shown that in the half-plane uE, $u > 0$ lines $L_1°$ and $L_2°$ have either one or three intersection points, and in the interval $0 < u \leqslant 1$ there are either two such points or they are altogether absent. In the former case one of these points is in the subsonic and the other in the supersonic region, or both lie in the supersonic region. The determination of possible disposition of intersection points is carried out in the manner described in Sects. 2 and 3 by superposing one of the intersection points with the point of minimum of curve $L_2°$ (for $S > S_*$). The integral curves of Eq. (2.1) ($\mathrm{Pr} \ll 1$) and of Eq. (3.4) ($\mathrm{Pr} \gg 1$), which describe the flow with a negative current lie above the straight line $L_1°$. Let us examine in detail the behavior of integral curves in the case in which in the interval $0 < u \leqslant 1$ there are two intersection points of lines $L_1°$ and $L_2°$. Let point A $(u^{(1)}, E^{(1)})$ lie in the supersonic region. The second intersection point B $(u^{(2)}, E^{(2)})$ is shown in Fig. 7 lying in the subsonic region and in Fig. 8 in the supersonic region. The integral curve which for $\varepsilon \to 0$ leaves the small neighborhood of point A with zero slope is denoted by I. The integral curves which leave with zero slope the neighborhood of lines $L_2°$ above point A (Figs. 7 and 8) or below point B (Fig. 8) define the structure of evolutionary shock waves with a continuous electric field at their front. In Fig. 8 the integral curve which for $\varepsilon \to 0$ leaves with zero slope the neighborhood of point B is denoted by II. For various electric fields within the limits

$$E^{(2)} < E < E^{(1)} \qquad (4.1)$$

the integral curves which in Fig. 7 are shown lying below curve I and in Fig. 8 between curves I and II pass for $\varepsilon \to 0$ through the small neighborhood of point A and, then, continue with a small positive slope up to the point of their intersection with the subsonic part of line L_2°.

Sections of these integral curves which lie above point A and below their intersection points with line L_2° for $\varepsilon \to 0$ run close to the line L_2° which corresponds to an inviscid flow. These integral curves define the structure of evolutionary shock waves with a discontinuity of the electric field at their front. In this case the parameters at point A are defined by the relationship $E^{(1)} + R_q u^{(1)} = 0$ (or in dimensional form by $u_I^* + bE_I^* = 0$) and correspond to the state ahead of the wave front. To determine the state behind the front of such shock waves it is necessary to specify one of the parameters there, in particular, the electric field may be specified within the limits given by (4.1). The structure of similar shock waves was considered in [4] for the case of a small interaction parameter. The integral curve I defines the structure of an evolutionary shock wave, when the field behind its front is $E_{II}^* = E_I^* = -u_I^* / b$, while the integral curve II (Fig. 8) defines the structure of an evolutionary shock wave, when the field ahead is $E_I^* = -u_I^* / b$ and behind it $E_{II}^* = -u^{(2)*} / b$ ($u^{(2)*}$ is the dimensional value of $u^{(2)}$). Behind the front of such shock wave the velocity is subsonic and equal to the smaller root of equation $L_2(u, E^{(2)}) = 0$.

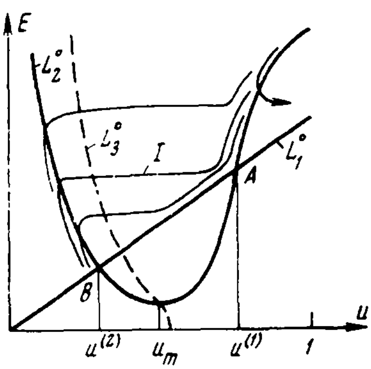

Fig. 7

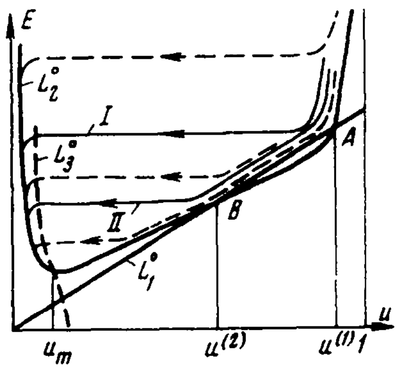

Fig. 8

Besides the integral curve II which for $\varepsilon \to 0$ leaves the small neighborhood of point B with zero slope (Fig. 8) there is a set of close-lying integral curves leaving the small neighborhood of that point with a slope equal to that of line L_2°. Below point B part of these integral curves run in close proximity to line L_2° Such integral curves define the structure of shock waves, when the parameters ahead and behind their front are related by $u_k^* + bE_k^* = 0$ ($k = \mathrm{I}, \mathrm{II}$) and the velocity behind and ahead of the wave front are supersonic and equal to the greater root of equation $L_2(u, E^{(2)}) = 0$. Waves of this kind are evolutionary. The number of equations defining the state at their front is equal six. They are: the laws of mass, momentum and energy conservation, Ohm's law, and the two conditions $u_k^* + bE_k^* = 0$ ($k = \mathrm{I}, \mathrm{II}$). For a wave to be evolutionary it is necessary that the number of low intensity waves radiating from the discontinuity be equal five (one less than the number of equations at the discontinuity front). Downstream of the discontinuity there are in this case three short wave highfrequency perturbations in the form of one entropic and two sonic waves which propagate at velocities u_{II}^* and $u_{II}^* \pm a_{II}^*$. respectively. All parameters, with the exception of the electric field, vary in these perturbations. An ion entropic wave, in which only the charge and current densities vary, propagates upstream of the discontinuity. Perturbations of the electric field δE_I^* and δE_{II}^* to the left and right of discontinuity, respectively, propagate at infinite velocity (the wave is of infinite length), while the charge density remains unperturbed [5]. Let us assume that δE_I^* (or δE_{II}^*) is specified, which corresponds to a wave of this kind reaching the discontinuity. The quantity δE_{II}^* (or δE_I^*) is then determined by the six relationships at the discontinuity stated above together with the amplitudes of the remaining four propagating waves and the parameter characterizing the perturbation of the discontinuity velocity.

In the case when the relation $u_k^* + bE_k^* = 0$ is fulfilled only ahead of the discontinuity front $(k = \text{I})$ or only behind the discontinuity front $(k = \text{II})$, the number of relationships at the discontinuity is equal five, and the number of propagating waves from the discontinuity is also less by one since the gas velocity behind the discontinuity front, in these cases, is subsonic.

If only one intersection point of lines L_1° and L_2° exists and is located in the supersonic region, and if ahead of the wave front the gas velocity is $u_m < u < u^{(1)}$ ($u^{(1)}$ is the velocity at the intersection point), then there exist shock waves which have a structure. At the front of such waves the field is continuous. If the unique intersection point of lines L_1° and L_2° lies in the subsonic region there are no integral curves which correspond to shock wave structure.

An error was made in the derivation of the simplified equation of the shock adiabatic curve in Sect. 6 of [1]. The term $\theta_1 \vartheta V^2$, which for $V \sim 1$ is of the same order of magnitude as the retained terms of Eq. (6.2), was neglected here. Consequently, Eq. (6.3) and its corollaries are incorrect.

The dependence of possible states behind the shock wave front on the state ahead of it in the uE-plane is analyzed in the present paper.

The authors thank A. G. Kulikovskii for discussing this subject and for his valuable advice.

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Translated by J. J. D.